**Weak Disorder**

Let’s do a few more examples, wanna get some GF’s in real position space.

**Self Energy and Green’s function to first order** **using δ potential**

To be concrete, we will use choose a particular form for the potential in our second order evaluation of the self energy.



This has the unfortunate consequence of making blow up – because the Fourier Transform doesn't go to 0, but is finite. This is physically saying that the energy shift due to a delta function potential is infinite? Well at least this indicates that our perturbative approach doesn’t work in this case. Anyway, we have …



(where we changed variables to k’ = k + q) So as we can see, the real part is divergent, regardless of the singularity, as the integral goes as ρ(ε)/ε ~ 1/√ε in 3D, which isn’t integral obviously. But we’ll just ignore this since as argued above, if we had a more realistic, i.e., damped V(q), we would find that the real part is exceedingly small compared to the imaginary part. And speaking of, the imaginary part gives,

,

Where ρ(x) is the free particle (that is with respect to the unperturbed system) density of states. Observe that when ω < -μ, ρ(ω+μ) = 0, and so τ-1(ω < -μ) = 0 too. Note how the lifetime of the state – roughly the uncertainty in its lifetime - is inversely proportional to the density of states, which gives the spread in energy about a single state, or sort of its energy uncertainty. Now let’s look at the retarded Green’s function. Using our work above, we can see that:



We wish to reconstruct the Green’s function in position space. So we take the inverse Fourier transform,



Since the integrand goes as trig/k the integral is finite. Since r is positive, we need only close the contour in the u.h.p., which is lucky for us since that’s where the pole is. Note however, that if we’re interested in ω < -μ, then τ-1(ω) = 0, and we’ll have to reinsert the +i0+ factor we’ve left out from the iωn → ω + i0+ replacement. In either case, we’ll have a pole in the u.h.p. and in the l.h.p. which will factor in the following manner.



Where  is in the u.h.p. Using the residue theorem, we find,



We can see that if the scattering time were infinite, we’d just have are usual spherical waves, so that if we planted a particle at point r, with energy, ω, it would propagate forward/outward in a spherical wave fashion, with momentum k. But reintroducing the scattering time, we can get an idea of the length scale over which the spherical wave will be damped. We note that it is the presence of the self energy’s non zero scattering rate which causes us to have an exponentially damped spherical wave. Let’s work out what k+is. We have to solve:



This approximation to the root is fairly easily determined graphically (analytically too). Just consider the ray pointing to the coordinates of k2 and then recognize that (for small imaginary part) the ray pointing at half the angle and square root the length is roughly square root real part (since it caries most of length) and ½ tan of the angle (as this is roughly ½ the angle itself). In any event, this simplifies to:



So we see that G goes is approximately equal to,



And we see that the Green’s function is spatially damped on the order of ℓ(ω) (we have to square the Green’s function).

*Recalculation with linearization of spectrum near the Fermi Surface approximation*

Now for practice we’ll return to our calculation via another approach. We’ll linearize the spectrum near the Fermi surface and recalculate our position – energy space Green’s function. Let



Then,



It is argued that the integral will come mostly from k ~ kF, and that for larger or smaller values of k, the integrand will wash out their contributions. Therefore, we can extend the range of integration to .



This obviously isn’t what we got before. But…for small ω << μ, this reduces to . And then,



Which is our previous result. So linearizing the spectrum is O.K. as long as we stick to energies, ω close to the Fermi level. So mathematical moral of the story is that its usually OK to approximate a polynomial in the denominator (which doesn’t have a zero along the path of integration) linearly about its maximum.

**1D Real Space Green’s function to first order with δ potential**

Consider once again scattering of noninteracting electrons by the zero-range impurity potential. Ignoring the real part of the self-energy, the disorder-averaged Matsubara Green’s function is approximately.



Let’s find the real space Matsubara Green’s function. I’ll approximate τk as a constant τF.



Where k+ is the pole in the u.h.p.

*Recalculation with Fermi surface linearization approximation*

Next we’ll try the familiar trick of linearizing the spectrum and comparing that ‘easier’ calculation to our exact one here. So again the spectrum is roughly,



and so,



And now we’ll extend the integral to negative infinity since the main contribution will come from k ≈ kF (which ought to be fairly far from k = 0).



Now, to save us some work, we’ll note that this function is clearly a function only of the absolute value of x, and we’ll define,



So we can write,



Comparing with the exact result, we’d find that it is accurate generally speaking only for energies close to the Fermi level, and inverse lifetimes also small compared to the Fermi level. Next, we would like to calculate the (complex) time dependence of the Green’s function. So, constructing the inverse Fourier transform of out result,



Now we can see one advantage of the linearization scheme. Since it keeps the energies only to first order, it allows explicit summation as a geometric series. So working out Ωn, ωn explicitly in terms of n…



and,



So then,



and, recalling ω0 = 2π/β,



Collecting all the imaginary parts. Now let’s calculate the number density for this one dimensional gas. The number density is simply,



So,



Let’s consider the small T limit,



So we have density oscillations? Interesting.

**1D Real Space (on half line) Green’s function to first order with δ potential**

Consider a non-interacting one-dimensional Fermi gas confined by an impenetrable barrier to the half-line x > 0 (with impurities). We would have some clues as to what to use for basis functions from solving the single particle Hamiltonian in the presence of the potential barrier. Normalized eigenfunctions are indeed,



Note that k is actually restricted to be positive because negative k is just the same eigenfunction (to w/i an unimportant multiplicative constant). So we would first off try to form the position annihilation operator as,



But we want to make contact with the free particle creation operator. So we just say,



The normalization is kind of screwy, and ψ(x) = -ψ(-x). So do we care about what is happening for x, k < 0? Anyway, we’ll now calculate the single particle Matsubara Green’s function on the half line.



So,



Next we note that since G is a function only of |x-x´|. So,



Now we can calculate the number density using our previous result for the 1D Matsubara Green’s function. We need to calculate,



This is (taking small x limit in the first one)



Expanding for small x, we’d have,



So we see that the density is zero at the boundary, as we’d expect certainly. This effect should be due to the action of the potential barrier, and not from the impurities themselves. Expanding for large x, we’d have,



And here we see ‘Friedel’ oscillations of the number density. Finally we take the limit T → 0.



Well, this does go to zero as x 🡪 0. So that’s good.